8 Discrete Random Variables

DD

Intuitively, to tell whether a random variable is discrete, we simply consider the possible values of the random variable. If the random variable can be ~~is~~ limited to only a finite or countably infinite number of possibilities, then it is discrete.

Example 8.1. Voice Lines: A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. [15, Ex 3-1]

Definition 8.2. A random variable X is said to be a ***discrete random variable*** if there exists a countable number of distinct real numbers x_k such that D1

$$\sum_k P[X = x_k] = 1. \tag{11}$$

D2 In other words, X is a discrete random variable if and only if X has a countable support.

Example 8.3. For the random variable N in Example 7.8 (Three Coin Tosses), *H

The possible values are 0, 1, 2, 3

The collection of possible values $\{0,1,2,3\}$ is finite. So, the RV is discrete.

For the random variable S in Example 7.9 (Sum of Two Dice),

Example 8.4. Toss a coin until you get a H. Let N be the number of times that you have to toss the coin.

The possible values are 1, 2, 3, 4, ...

The collection of possible values $\{1,2,3\ldots\} = \mathbb{N}$ is countably infinite. So the RV is discrete.

8.5. Although the support S_X of a random variable X is defined as any set S such that $P[X \in S] = 1$. For discrete random variable, S_X is usually set to be $\{x : P[X = x] > 0\}$, the set of all "possible values" of X.

Definition 8.6. Important Special Case: An ***integer-valued random variable*** is a discrete random variable whose x_k in (11) above are all integers.

8.7. Recall, from 7.21, that the ***probability distribution*** of a random variable X is a description of the probabilities associated with X.

For a discrete random variable, the distribution can be described by just a list of all its possible values $(x_1, x_2, x_3, \ldots)$ along with the probability of each:

$$(P[X = x_1], P[X = x_2], P[X = x_3], \ldots, \text{ respectively}).$$

In many cases, it is convenient to express the probability in the form of a formula. This is especially useful when dealing with a random variable that has infinitely many outcomes. It would be tedious to list all the possible values and the corresponding probabilities.

8.1 PMF: Probability Mass Function

uppercase

Definition 8.8. When X is a discrete random variable satisfying (11), we define its ***probability mass function*** (pmf) by[32]

$p_X(3) = P[X=3]$

$p_X(5) = P[X=5]$

$$p_X(x) = P[X = x].$$

lowercase

subscript indicates the name of the RV

- Sometimes, when we only deal with one random variable or when it is clear which random variable the pmf is associated with, we write $p(x)$ or p_x instead of $p_X(x)$.

- The argument (x) of a pmf ranges over *all real numbers*. Hence, the pmf is (and should be) defined for x that is not among the x_k in (11) as well. In such case, the pmf is simply 0. This is usually expressed as "$p_X(x) = 0$, otherwise" when we specify a pmf for a particular random variable.

[32]Many references (including [15] and `MATLAB`) does not distinguish the pmf from another function called the probability density function (pdf). These references use the function $f_X(x)$ to represent both pmf and pdf. We will *NOT* use $f_X(x)$ for pmf. Later, we will define $f_X(x)$ as a probability density function which will be used primarily for another type of random variable (continuous RV).

- The pmf of a discrete random variable X is usually referred to as its ***distribution***.

Example 8.9. Continue from Example 7.8. N is the number of heads in a sequence of three fair coin tosses.

$$p_N(n) \equiv P[N=n] = \begin{cases} 1/8, & n=0,3, \\ 3/8, & n=1,2, \\ 0, & \text{otherwise.} \end{cases}$$

$P[N=0] = 1/8$

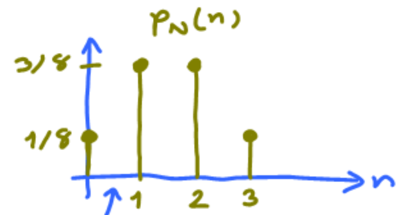
$P[N=1] = 3/8$

$P[N=2] = 3/8$

$P[N=3] = 1/8$

$P[N=n] = 0$, $n \notin \{0,1,2,3\}$

sketch of pmf:



Remark: The "0, otherwise" part is usually not drawn.

8.10. Graphical Description of the Probability Distribution: Traditionally, we use ***stem plot*** to visualize p_X. To do this, we graph a pmf by marking on the horizontal axis each value with nonzero probability and drawing a vertical bar with length proportional to the probability.

8.11. Any pmf $p(\cdot)$ satisfies two properties:

(a) $p(\cdot) \geq 0$

$\sum_x p_X(x) = 1$

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x.

When you are asked to verify that a function is a pmf, check these two properties.

8.12. Finding probability from pmf: for "any" subset B of $\mathbb{R}$, we can find

$$P[X \in B] = \sum_{x_k \in B} P[X = x_k] = \sum_{x_k \in B} p_X(x_k).$$

In particular, for integer-valued random variables,

$$P[X \in B] = \sum_{k \in B} P[X = k] = \sum_{k \in B} p_X(k).$$

ECS 315: In-Class Exercise # 10

Date: _ _ / _ _ / 2018			
Name	ID (last 3 digits)		

Instructions

1. Separate into groups of no more than three persons. **The group cannot be the same as any of your former groups.**
2. **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

$\rightarrow P(\{\omega\}) = \frac{1}{6}$ for $\omega = 1,2,3,\dots,6$

1. Consider a random experiment in which you roll a six-sided fair dice (whose faces are numbered 1-6). We define the following random variables from the outcomes of this experiment:

$$X(\omega)=\omega \quad \text{and} \quad Y(\omega)=1+\big((\omega-2)(\omega-3)(\omega-5)(\omega-8)\big).$$

a. Find $P[X=5]$.

$X(\omega) = 5$ when $\omega = 5$ $\Rightarrow P[X=5] = P(\{5\}) = \frac{1}{6}$

b. Find $P[Y=1]$.

$Y(\omega) = 1$ when $1+\big((\omega-2)(\omega-3)(\omega-5)(\omega-8)\big) = 1$

not in Ω

$\omega = 2, 3, 5, \cancel{8}$

$\Rightarrow P[Y=1] = P(\{2\}) + P(\{3\}) + P(\{5\}) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{1}{2}$

$P(\{\omega\}) = \frac{1}{10}$ for $\omega = 1,2,\dots,10$

2. Consider a random experiment in which you roll a 10-sided fair dice (whose faces are numbered 0–9). Define a random variable Z from the outcomes of this experiment by

$$Z(\omega)=(\omega-7)^2.$$

a. Find $P[Z=4]$.

$Z(\omega) = 4$ when $(\omega-7)^2 = 4$

$\omega = 7 + (\pm 2) = 5$ or 9

$\Rightarrow P[Z=4] = P(\{5\}) + P(\{9\}) = \frac{1}{10}+\frac{1}{10} = \frac{1}{5}$

b. Find $P[Z>20]$.

Method 1:

$Z(\omega) > 20$ when $(\omega-7)^2 > 20$

$\cancel{\omega > \underbrace{7+\sqrt{20}}_{\approx 11.4721}}$ or $\omega < \underbrace{7-\sqrt{20}}_{\approx 2.5279}$

none of the ω in Ω satisfies this condition

$\omega = 0, 1, 2$

same

$\Rightarrow P[Z>20] = P(\{0\}) + P(\{1\}) + P(\{2\}) = \frac{3}{10}$

Method 2: Because Ω is not large, it is possible to find $X(\omega)$ for all ω.

ω	$\omega-7$	$(\omega-7)^2$
0	−7	49
1	−6	36
2	−5	25
3	−4	16
4	−3	9
5	−2	4
6	−1	1
7	0	0
8	1	1
9	2	4

>20

Additional Example from the in-class exercise:

we roll a fair six-sided dice and define

$\Omega = \{1, 2, \ldots, 6\}$

$P(\{\omega\}) = 1/6 \quad \omega = 1, 2, \ldots, 6$

$$Y(\omega) = 1 + \left((\omega-2)(\omega-3)(\omega-5)(\omega-8)\right)$$

Find $p_Y(y) \equiv P[Y=y]$

ω	$Y(\omega)$
1	57
2	1
3	1
4	9
5	1
6	-23

$$p_Y(y) = \begin{cases} & \\ 0, & \text{otherwise} \end{cases}$$

Because we want to find the pmf Y, we need to calculate $P[Y=y]$ for any y.

Not that many!

If y is an impossible value, then $P[Y=y] = 0$.

Here, the only possible values of the RV Y are $\underset{y_1}{-23}, \underset{y_2}{1}, \underset{y_3}{9}, \underset{y_4}{57}$.

Y is a discrete RV because it has four possible values.

four — finite

$p_Y(-23) \equiv P[Y=-23] = P(\{6\}) = \frac{1}{6}$

$p_Y(1) \equiv P[Y=1] = P(\{2,3,5\}) = \frac{3}{6} = \frac{1}{2}$

$p_Y(9) \equiv P[Y=9] = P(\{4\}) = \frac{1}{6}$

$p_Y(57) \equiv P[Y=57] = P(\{1\}) = \frac{1}{6}$

$$p_Y(y) = \begin{cases} 1/6, & y = -23, 9, 57, \\ 1/2, & y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$P[Y=y_1] + P[Y=y_2] + P[Y=y_3] + P[Y=y_4] = \frac{1}{6} + \frac{1}{2} + \frac{1}{6} + \frac{1}{6} = 1$$

8.13. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfy the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

Example 8.14. Back to Example 7.7 where we roll one dice.

- The "important" probabilities are

$$P[X=1] = P[X=2] = \cdots = P[X=6] = \frac{1}{6}$$

- In tabular form:

Dummy variable

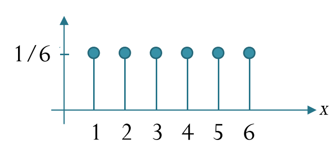
x	$P[X=x]$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

- **Probability mass function (PMF)**:

$$p_X(x) = \begin{cases} 1/6, & x = 1,2,3,4,5,6, \\ 0, & \text{otherwise.} \end{cases}$$

  - In general, $p_X(x) \equiv P[X=x]$

- Stem plot:

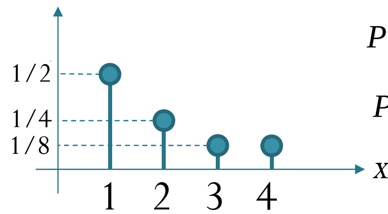


Suppose we want to find $P[X > 4]$.

Steps	For this example…
Find the support of X.	The support of X is $\{1,2,3,4,5,6\}$.
Consider only the x inside the support. Find all values of x that satisfy the condition(s).	The members which satisfies the condition ">4" is 5 and 6.
Evaluate the pmf at x found in the previous step.	The pmf values at 5 and 6 are all 1/6.
Add the pmf values from the previous step.	Adding the pmf values gives 2/6 = 1/3.

Example 8.15. Consider a RV X whose $p_X(x) = \begin{cases} 1/2, & x = 1, \\ 1/4, & x = 2, \\ 1/8, & x \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$

stem plot:



$$P[X=2] = p_X(2) = \frac{1}{4}$$

$$P[X>1] = p_X(2) + p_X(3) + p_X(4) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Example 8.16. Suppose a random variable X has pmf

$$p_X(x) = \begin{cases} c/x, & x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

x	$p_X(x)$
1	c
2	$c/2$
3	$c/3$

(a) The value of the constant c is

For any pmf, $\sum_x p_X(x) = 1$

$c + \frac{c}{2} + \frac{c}{3} + 0 = 1$

$c = \frac{1}{1 + \frac{1}{2} + \frac{1}{3}} = \frac{6}{11}$

(b) Sketch its pmf

(c) $P[X = 1] = p_X(1) = c = \frac{6}{11}$

(d) $P[X \geq 2] = p_X(2) + p_X(3) = \frac{c}{2} + \frac{c}{3} = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}$

(e) $P[X > 3] = 0$

None of the x values in the support satisfies the condition "$x > 3$".

8.17. Any function $p(\cdot)$ on $\mathbb{R}$ which satisfies

(a) $p(\cdot) \geq 0$, and

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x

is a pmf of some discrete random variable.

8.2 CDF: Cumulative Distribution Function

Definition 8.18. The ***(cumulative) distribution function (cdf)*** of a random variable X is the function $F_X(x)$ defined by

$$F_X(x) = P[X \leq x].$$

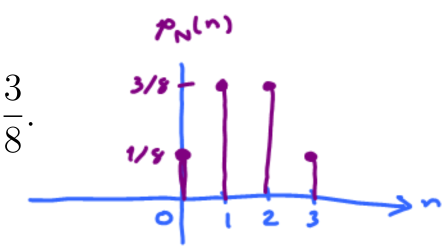
- The argument (x) of a cdf ranges over all real numbers.
- From its definition, we know that $0 \leq F_X \leq 1$.
- Think of it as a function that collects the "probability mass" from $-\infty$ up to the point x.

8.19. From pmf to cdf: In general, for any discrete random variable with possible values $x_1, x_2, \ldots$, the cdf of X is given by

$$F_X(x) = P[X \leq x] = \sum_{x_k \leq x} p_X(x_k).$$

Example 8.20. Continue from Examples 7.8, 7.17, and 8.9 where N is defined as the number of heads in a sequence of three coin tosses. We have

$$p_N(0) = p_N(3) = \frac{1}{8} \text{ and } p_N(1) = p_N(2) = \frac{3}{8}.$$

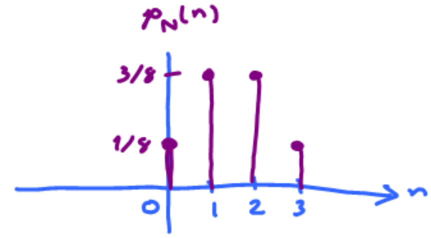


(a) $F_N(0) \equiv P[N \leq 0] = p_N(0) = \frac{1}{8}$

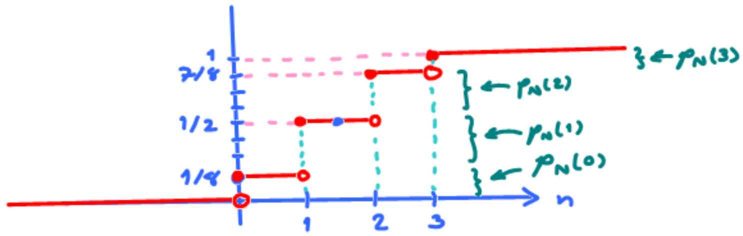
(b) $F_N(1.5) \equiv P[N \leq 1.5] = p_N(0) + p_N(1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$

The values of N that is "≤1.5" are 0, 1

"pmf → cdf"



(c) Sketch of cdf $F_N(n) \equiv P[N \le n]$



8.21. Facts:

- For any discrete r.v. X, F_X is a right-continuous, ***staircase*** function of x with jumps at a countable set of points x_k.
- When you are given the cdf of a discrete random variable, you can derive its pmf from the locations and sizes of the jumps. If a jump happens at $x = c$, then $p_X(c)$ is the same as the amount of jump at c. At the location x where there is no jump, $p_X(x) = 0$.

"cdf → pmf"

Example 8.22. Consider a discrete random variable X whose cdf $F_X(x)$ is shown in Figure 15.

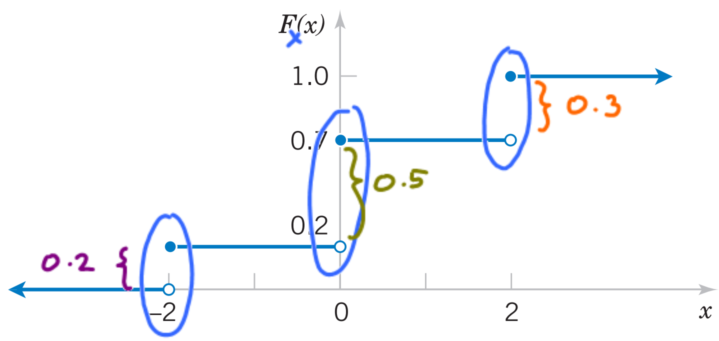


Figure 15: CDF for Example 8.22

Determine the pmf $p_X(x)$.

$$p_X(x) = \begin{cases} 0.2, & x = -2, \\ 0.5, & x = 0, \\ 0.3, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 7 (M2013)**.** (8 pt) The cdf of a random variable X is plotted in Figure 1.1.

"cdf → pmf"

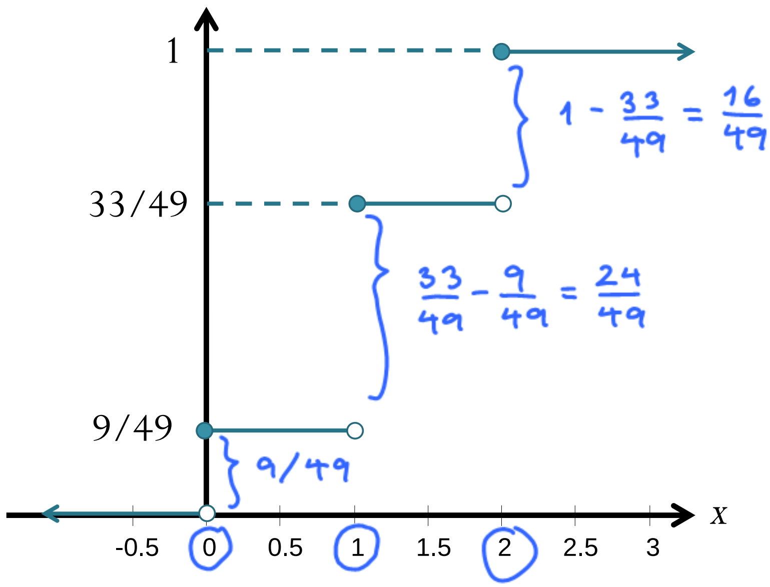
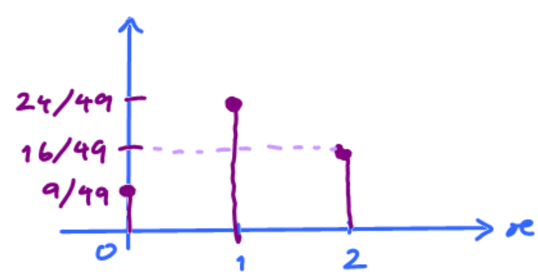


Figure 1.1: CDF of X for Problem 7

(a) (4 pt) Find and carefully plot the pmf $p_X(x)$.

$$p_X(x) = \begin{cases} 9/49, & x = 0, \\ 24/49, & x = 1, \\ 16/49, & x = 2, \\ 0, & \text{otherwise} \end{cases}$$



(b) (2 pt) Find $P[X > 1]$.

"2" is the only possible value of X that satisfies "> 1".

Method 1: $P[X>1] = p_X(2) = 16/49$.

Method 2: $P[X>1] = P(\underbrace{[X>1]}_{A}) = P(A) = 1 - P(A^c) = 1 - P([X>1]^c)$

$= 1 - P[X \le 1] = 1 - F_X(1) = 1 - \frac{33}{49} = \frac{16}{49}$

8.23. Characterizing[33] properties of cdf:

CDF1 F_X is non-decreasing (monotone increasing)

$\equiv$ if $a < b$, then $F_X(a) \le F_X(b)$

CDF2 F_X is right-continuous (continuous from the right)

$$\equiv \lim_{y \searrow x} F_X(y) = F_X(x)$$

$$\equiv \lim_{\substack{h>0 \\ h \to 0}} F_X(x+h) = F_X(x)$$

$$\equiv F_X(x^+) = F_X(x)$$

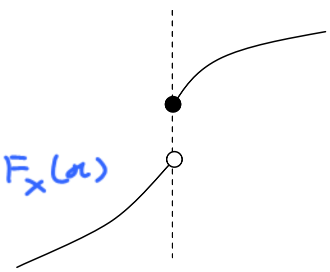


Figure 16: Right-continuous function at jump point

CDF3 $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

8.24. For discrete random variable, the cdf F_X can be written as

$$F_X(x) = \sum_{x_k} p_X(x_k) u(x - x_k),$$

where $u(x) = 1_{[0,\infty)}(x)$ is the unit step function.

[33] These properties hold for any type of random variables. Moreover, for any function F that satisfies these three properties, there exists a random variable X whose CDF is F.